

Factorization in Dual Models and Functional Integration in String Theory.

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This article contains a summary of the author's contributions, one in collaboration with K. Bardakçi, to dual models and string theory prior to the mid-seventies. Other workers' contributions, during and subsequent to this period, are mentioned in order to relate our work to the general development of the subject.

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1 Introduction

In accordance with the scope of the present volume, this article will mainly be concerned with work during the relevant period in which the author was involved, with a minimum of technicalities. Other contributions will be mentioned briefly in order to indicate the place of our work in the general development. Details of these contributions as will not be given, as they are adequately covered in other articles in the volume. We emphasize that the contributions are chosen in order to provide continuity with the contributions in which we were involved, and not for their importance. There will be many uncited contributions of greater importance than those cited. The historical development will be emphasized.

As every reader of this volume probably knows, string theory can be traced back to the discovery by Veneziano [1] of a formula for a four-point scattering amplitude (two incoming and two outgoing particles) with narrow resonances and rising Regge trajectories in both channels. Another such formula, for what are now known as closed strings, was proposed by Virasoro [2] and written as an integral by Shapiro [3]. These formulas were then extended to five-point amplitudes independently by Bardakçi and Ruegg and by Virasoro [4] and then to N -point amplitudes independently by Bardakçi and Ruegg; Chan and Tsou; Goebel and Sakita; and Koba and Nielsen [5] [6]. For simplicity most of the treatment in the remainder of this article will be for open strings; the closed-strings formulas will be very similar. The most convenient expression for the general open-string N -point amplitude is the following, which we quote here for future reference

$$A = \int d^{n-3}z |(z_b - z_a)(z_c - z_b)(z_c - z_a)| \prod_{i>j} |z_i - z_j|^{-2p_i p_j} \quad (1.1)$$

The subscript i refers to the i -th particle, with momentum p_i . The variables z_i are ordered cyclically along the real line. Three of them, which we have denoted by z_a, z_b, z_c are arbitrarily chosen and held at arbitrary fixed values; we integrate over the other $n - 3$ subject to the condition of cyclic ordering. By making use of the invariance of the amplitude (1.1) under a projective transformation of the variables

$$z' = \frac{az + b}{cz + d}, \quad (1.2)$$

we can easily see that the integral is independent of the choice and values of the constant z 's. We have written the formula for the case where the external particles are tachyons with μ^2 , the square of their mass, equal to -1 in units of the slope of the Regge trajectories; the formula for the general case is slightly more complicated. At this stage of the development there is no reason for this choice of the mass, but later on we shall see that it is necessary in a consistent model.

The formula for the closed-string amplitude is similar but not identical to (1.1), except that the integral is now over the whole complex plane, and the variables are not restricted by cyclic ordering.

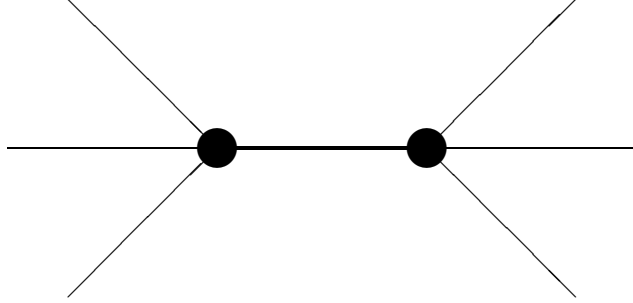


Figure 1: Factorization of scattering amplitudes at a pole.

If the variable $p_i \cdot p_j$ is sufficiently large, the above integral diverges when $(z_i - z_j)$ approaches zero. One can obtain finite results by starting from an algebraically smaller value of $p_i \cdot p_j$ and analytically continuing or, equivalently, by integrating by parts and dropping the end-point contribution. At the time, such a procedure was justifiable, since one was simply trying to obtain an S-matrix element with the correct analytic properties. We shall return to this point later.

2 Factorization

In order that the scattering amplitudes should together form a consistent S -matrix, they must satisfy a requirement known as factorization (the narrow-resonance equivalent of unitarity). The residue at the pole of an amplitude as a function of s , the square of the energy, should consist of a number of terms, one for each excited particle, with the square of its mass equal to s . Each term has to be the product of two factors, one depending only on the number of incoming particles and their momenta, the other on the corresponding variables of the outgoing particles. The factorizability of the amplitude (1.1) was shown independently by Bardakçi and Mandelstam and by Fubini and Veneziano [7]. In Figure 1, we define the variables corresponding to the particles on the left as z_1, \dots, z_j , and those corresponding to the particles on the right as z_{j+1}, \dots, z_N . We fix the variables z_1, z_{j+1} and z_N at the values 0, 1, and ∞ . We then define new variables

$$x_i = \frac{z_i}{z_j}, \quad 1 \leq i \leq j \quad z = z_j, \quad y_k = \frac{z_{j+1}}{z_k}, \quad j+1 \leq k \leq N \quad (2.1)$$

The formula (1.1) now becomes

$$A = \int dz dx_2, \dots, dx_{j-1} dy_{j+2}, \dots, dy_{N-1} z^{-s-1} I_1 I_2 \prod_{i=1}^j \prod_{k=j+1}^N (1 - zx_i y_k)^{-2p_i \cdot p_k}, \quad (2.2)$$

where

$$I_1 = \prod_{i=1}^j \prod_{k=i+1}^j (x_k - x_i)^{-2p_i \cdot p_k}, \quad I_2 = \prod_{i=j+1}^N \prod_{k=i+1}^N (y_i - y_k)^{-2p_i \cdot p_k},$$

$$0 = x_1 \leq x_2 \leq \dots \leq x_j = 1, \quad 1 = y_{j+1} \geq \dots \geq y_{N-1} \geq y_N = 0,$$

and s is the square of the energy of the intermediate state in Figure 1.

To calculate the residue R at the pole in (2.2) when $s = n$, where n is a non-negative integer, we first have to integrate by parts n times as described above. We then find

$$R = -\frac{1}{n!} \int dx_2 \dots dx_{j-1} dy_{j+2} \dots dy_{N-1} I_1 I_2 \frac{\partial^n}{\partial z^n} \prod_{i=1}^j \prod_{k=j+1}^N (1 - zx_i y_k)^{-2p_i \cdot p_k} \Big|_{z=0} \quad (2.3)$$

By expanding the last factor in (2.3) in a power series and taking the coefficient of z^n , we find that R can be written as the sum of terms of the form

$$c \int dx_2 \dots dx_{j-1} dy_{j+2} \dots dy_{N-1} I_1 I_2 \sum_r \prod_{i=1}^j \sum_{k=j+1}^N (2p_i \cdot p_k x_i^r y_k^r)^{\lambda_r}, \quad (2.4)$$

where

$$\sum_r r \lambda_r = n \quad (2.5)$$

Equations (2.4) and (2.5) show that the residue is indeed a finite sum, each term corresponding to a state of an assembly of d -vector simple harmonic oscillators. The integer r labels the oscillator, the level spacing of the r th oscillator is proportional to r , and the energy of the ground state is equal to zero. In the term shown the r th oscillator is in level λ_r . (More accurately, we should expand the dot product in (2.4) and then expand the λ_r th power of the sum; each term will correspond to a state where the r th oscillator is in level λ_r , with the energy divided among the components in the obvious way.) We have thus shown that the amplitude (1.1) does factorize as required, and the states of the system correspond to the assembly of simple harmonic oscillators described above.

The factorization properties of the N -point amplitude were used independently by Fubini, Gordon and Veneziano, by Nambu and by Susskind [8] to construct an operator formalism for the dual model, the operators being the creation and destruction operators for the above assembly of oscillators. These authors were able to construct the N -point amplitude in terms of such operators; the factorization properties were then obvious. In fact, the easiest way to describe dual models was seen to be to start from the operator formalism, thus reversing the historical order.

3 Further Developments

Two very similar new dual models were proposed by Ramond [9] (R), initially for free fermions, and by Neveu and Schwarz [10] (NS) for bosons; the latter model was slightly reformulated by Neveu, Schwarz and Thorn. These models both had a series of anti-commuting d -vector harmonic-oscillator operators (b -operators) in addition to the a -operators mentioned at the end of the last section. The difference was that the Ramond operators had integral mode numbers, the zero-mode operators being interpreted as γ -matrices, while the NS operators had half-integral mode numbers.

It was suggested by Kikkawa, Sakita and Virasoro [11] (KSV) that the amplitude (1.1) might be the Born term of a perturbation series, and they proposed a form for the n -loop term. The simplest one-loop term was calculated independently from unitarity by Amati, Bouchiat and Gervais, by Bardakçi, Halpern and Shapiro and by Kikkawa, Sakita, Veneziano and Virasoro [12]. More general terms were calculated by Kaku and Thorn [13]; an improvement to their calculation was made by Gross, Neveu, Scherk and Schwarz (GNSS) [14]. The N -loop amplitude was calculated independently by Alessandrini, by Kaku and Yu and by Lovelace [15]; their work was further developed by Alessandrini and Amati [16]. Lovelace and Alessandrini, in common with all these workers, based their calculation on the operator formalism mentioned above, but they pointed out that their calculation could be understood in terms of amplitudes associated with n -hole Riemann surfaces. Their work was motivated by ideas suggested by Nielsen [25] in connection with the recently proposed string interpretation of dual models, and the associated analogue model of Fairlie and Nielsen [17]. All these calculations, however, had ghosts,¹ which we shall treat shortly, circulating in the loops, and they were therefore incorrect. We mention them because of their relation to later work.

GNSS showed that certain one-loop amplitudes possessed cuts not associated with unitarity. Such cuts were found independently by Frye and Susskind [18] in a calculation based on the analogue model. Lovelace [19], making a reasonable conjecture about the amplitude with ghosts eliminated, showed that these cuts became poles if $d=26$. It thus appeared that the model was inconsistent unless $d=26$, a feature which caused considerable amusement among sceptics at the time.

Since the products in Eq. (2.4) are d -vector products, or, equivalently, the a -operators in the operator formalism are d -vectors, the states corresponding to the time component will be ghosts, i. e., negative-metric states. The model will not therefore be physically acceptable unless the ghosts can be eliminated. In the case where the ground state was a tachyon with $\mu^2 = -1$, Virasoro [20] found an infinite series of operators which, when applied to any particle in the operator formalism, gave a "spurious" state, i. e., a state which would not appear as an intermediate state in Figure 1 (The Virasoro operators satisfy the algebra of two-dimensional conformal transformations with a central charge.)

¹By ghost we mean any particle with a negative metric. The Faddeev-Popov ghosts, of course, were not involved in dual models or string theory at this stage.

The physical Hilbert space was therefore the space orthogonal to all the spurious states, and the hope was that this Hilbert space would be ghost-free. Del Giudice, DiVecchia and Fubini [21] (DDF) found a set of "transverse" positive-definite simple-harmonic oscillators, with $d - 2$ components, which commuted with the Virasoro operators and which could therefore be used to construct a ghost-free subspace of the Hilbert space, or possibly the whole Hilbert space. Finally Brower [22] showed that the DDF operators, together with a new set of "longitudinal" operators, could be used to construct the entire physical Hilbert space, and that the Hilbert space so constructed was ghost-free provided $d \leq 26$.

Brower showed that the situation was particularly simple if $d = 26$, when the longitudinal operators produced null states which did not give rise to poles in Figure 1, so that the physical Hilbert space constructed from the DDF operators without the longitudinal operators was sufficient. The string models, as presently formulated, cannot incorporate the longitudinal modes, and, in fact, no procedure is known for allowing only the particles in the physical Hilbert space to circulate in loops if there are transverse and longitudinal modes². We thus again obtain the condition $d=26$ as a condition for consistency. We shall denote $d=26$ as the critical dimension. Goddard and Thorn [23] have given a simpler proof of ghost elimination if $d=26$.

The ghost-free physical Hilbert space for the NS model can be constructed in a very similar way [24] [23]. Here the square of the mass of the ground state must be equal to minus one-half of the slope of the Regge trajectory, and the critical dimension is $d=10$.

It was observed independently by Nambu, Nielsen and Susskind [25] that the oscillators of dual models could be considered as the modes of vibration of a string. Nambu and Goto [26] took the action for the world sheet of the string moving in time to be the area of the sheet. Goddard, Goldstone, Rebbi and Thorn [27] (GGRT) have quantized the noninteracting dual string. They first considered a classical string and used light-cone coordinates

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{d-1}), \quad X^- = \frac{1}{\sqrt{2}}(X^0 - X^{d-1}), \quad X^i \quad 1 \leq i \leq d-2 \quad (3.1)$$

They then made a choice of coordinates σ, τ on the string world-sheet. Making such a choice is equivalent to fixing a gauge in a gauge theory. They showed that they could make a choice such that $\tau=X^+$, that the momentum per unit length P^+ is a constant, that X^- is determined in terms of the X^i 's, and that the Nambu-Goto Lagrangian is

$$\mathcal{L} = \frac{1}{4\pi} \left\{ \left(\frac{\partial X^i}{\partial \tau} \right)^2 - \left(\frac{\partial X^i}{\partial \sigma} \right)^2 \right\} \quad (3.2)$$

The condition of constant P^+ -momentum per unit length means that the "length" of the string is proportional to the total value of p^+ . The strings satisfy the boundary condition that the slope, $\frac{\partial X^i}{\partial \sigma}$, is zero at the ends. The GGRT choice of

²But such a model may possibly be equivalent to the Polyakov noncritical string

co-ordinates is not, of course, Lorentz covariant, and the authors construct generators for the nontrivial Lorentz transformations. The transformations M_{i+} , which change X^+ , are the difficult ones, since the condition $\tau = X^+$ has to be reestablished. This is done by making a pseudo-conformal transformation of the $\sigma - \tau$ co-ordinates, since the action is pseudo-conformally invariant. The Lorentz generators for the non-trivial transformations are trilinear.

Quantization of the model with the Lagrangian (3.2) is straightforward. The Hamiltonian is normal ordered, and a constant term μ_0^2 is added. There is no problem with ghosts, since the Lagrangian only involves transverse co-ordinates. However, Lorentz invariance is nontrivial. The classical Lorentz generators do effect the required transformations, but anomalous terms appear in the commutator between transformations M_{i+} and M_{j+} when $i \neq j$. These terms cancel when $d - 2 = 24$ and $\mu_0^2 = -1$, which are thus the conditions for Lorentz invariance of the theory.

The Ramond and Neveu-Schwarz models were treated by Iwasaki and Kikkawa [28] in a similar way. As might be expected, the conditions for Lorentz invariance were $d - 2 = 8$ and $\mu_0^2 = -\frac{1}{2}$ (Neveu-Schwarz) or $\mu_0^2 = 0$ (Ramond).

4 Interacting Strings and Functional Integration

We now treat the subject of interacting strings by functional integration. The problem was initially considered by Gervais and Sakita [29]. The amplitudes constructed by them were not manifestly factorizable and the condition $d = 26$ was not evident in their work. The approach we shall describe [30] was based on the light-cone GGRT string just treated.

It was realized independently by several people that a theory of interacting strings could be constructed by allowing the free strings to split and join. The Hilbert space now consists of any number of strings. The free Lagrangian will be a sum of terms of the form (3.2), one for each string, together with an interaction which decreases or increases or decreases the number of strings by one. The interaction vertex function will be zero unless the position of the initial and final strings coincide. In terms of operators, the vertex function V will be the infinite product:

$$V = \prod_{\sigma, i} \delta\{X_f^i(\sigma) - X_i^i(\sigma)\} \quad (4.1)$$

Two strings can also meet at a point and recombine, but we shall not encounter this term in our treatment.

The string world sheet for a general interacting string process is shown in Figure 2. The horizontal axis is the light-cone time τ , the vertical axis is the length σ . The $d - 2$ transverse coordinates X^i are orthogonal to the paper. Thus three strings come in from $\tau = -\infty$, after a time the lower two join, and so on until we reach $\tau = +\infty$. Note that the X^i coordinates are discontinuous across the horizontal lines, since they are coordinates of different strings. We have not separated the strings in the diagram to illustrate the fact that the total "length" of all the strings, i. e., the total value of P^+ , is constant.

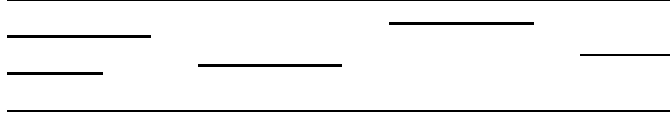


Figure 2: Interacting string diagram

For closed strings, the ends of all strings in Figure 2 are identified, so that the diagram consists of cylinders, each cylinder representing a closed string propagating in time. At the interaction points, two closed strings join to form one or one string splits into two.

The scattering amplitude associated with the process shown will be found by functional integration of the transverse coordinates $X^i(\sigma, \tau)$. To begin, we integrate separately over the parts of the world sheet associated with the different strings, with a vertex function at each interaction time. Since the vertex function requires the X^i 's to be continuous across the interaction times, we may simply integrate over the whole world sheet. The X^i 's are discontinuous over the horizontal lines of figure 2, and, at the horizontal lines, they will satisfy Neumann boundary conditions, namely that their derivatives orthogonal to the boundary are zero.

Before performing the functional integration, we continue analytically to Euclidean time. Thus Eq. (3.2) now becomes

$$\mathcal{L} = -\frac{1}{4\pi} \left\{ \left(\frac{\partial X^i}{\partial \tau} \right)^2 + \left(\frac{\partial X^i}{\partial \sigma} \right)^2 \right\} \quad (4.2)$$

After performing the functional integration, we integrate over the intermediate times in figure 2, which we denote by τ_1, \dots, τ_r . The scattering amplitude for unexcited incoming and outgoing particles is thus:

$$A = \mathcal{N} \int d\tau_1 \dots [d\tau_i] \dots d\tau_R \int \prod dX^i(\sigma, \tau) \times \exp \left\{ i \sum_r \frac{1}{\pi \alpha_r} p_r^i \int d\sigma X^i(\sigma, \tau_0) + \int d\sigma d\tau \mathcal{L}(\sigma, \tau) - \sum_r p_r^- \tau_0 \right\} \quad (4.3)$$

The factor \mathcal{N} is a normalization factor, which is a product of an overall normalization factor and a factor for each incoming or outgoing particle. By considering the special case of a free string, it can be shown to be unity. The first sum in the exponent is over all incoming and outgoing particles; it is the wave function for the zero mode of the state. The factor $\pi \alpha_r$ is the “length” of the r th string, and the integral is over the string. The time τ_0 is the time at the beginning or end of Figure 2 (large negative or large positive), depending on whether the particle is incoming or outgoing. The wave functions of the nonzero modes of the string have been omitted, since they will simply contribute to the normalization factor

\mathcal{N} if the particle is in its unexcited state. They must be included for general excited particles. The square brackets around τ_i indicate that one τ is omitted from the variables of integration, as the entire amplitude is invariant under time translation. The integration over this variable simply gives us conservation of energy. There are thus $N - 3$ variables of integration.

Since the functional integrand in (4.3) is a quadratic function of the X^i , the functional integral can be performed by standard methods in terms of the reciprocal of the Laplacian and its determinant. We define the Neumann function for the string world sheet by the equation:

$$\left\{ \left(\frac{\partial}{\partial \sigma} \right)^2 + \left(\frac{\partial}{\partial \tau} \right)^2 \right\} N(\sigma, \tau; \sigma', \tau') = 2\pi \delta(\sigma - \sigma') \delta(\tau - \tau') - 1/A \quad (4.4)$$

with Neumann boundary conditions, i. e., the σ derivatives of N must be zero at the horizontal lines of Figure 2. A is the area of the string diagram, with large finite initial and final times. The last term is to take into account the zero mode, since the inverse of the Laplacian operator can only be defined if the zero mode is excluded.³ Even though this term tends to zero as the initial and final times approach infinity it must be included, since we really have to take the functional integral with finite initial and final times, and then let the times approach $\pm\infty$.

On performing the functional integral, we obtain the result:

$$A = \int d\tau_1 \dots [d\tau_i] \dots d\tau_R \Delta^{(d-2)/2} \times \exp \left\{ \sum_{r,s} \frac{1}{\pi^2 \alpha_r \alpha_s} p_r^i p_s^i \int d\sigma d\sigma' N(\sigma, \tau_0; \sigma', \tau'_0) - \sum_r p_r^- \tau_0 \right\} \quad (4.5)$$

The factor Δ is $-1/2\pi$ times the determinant of the Laplacian on the string diagram. The first summation within the braces is now over all pairs of strings, and the σ -integral is over the string at $\tau = \pm\infty$. If the two strings are different (and at $\pm\infty$), the Neumann function will be independent of the position along the string, so that the σ integrals may be written $N(r, s)$, where r and s denote the positions of the r th and s th string. The integrals where r and s refer to the same string contribute normalization factors, independent of the shape of the string diagram. We denote the product of all these normalization factors by N . Furthermore, it is not difficult to show that the last term in the braces of equation (4.5) simply changes the $(d-2)$ -vector product $p_r^i p_s^i$ into a d -vector product. On inserting these changes in equation (4.5), we obtain our final result:

$$A = \int d\tau_1 \dots [d\tau_i] \dots d\tau_R \Delta^{(d-2)/2} N \exp \left\{ -2 \sum_{r>s} p_r \cdot p_s N(r, s) \right\}. \quad (4.6)$$

³We are here improving on our original treatment of the extra terms on the right of (4.4). See [31, page 170]. As will probably be evident in what follows, the final results are unaffected.

As we have been using the light-cone frame our approach, while manifestly unitary in a positive-definite Hilbert space, is not manifestly Lorentz invariant. (While the momentum-dependent factor in equation (4.6) is Lorentz covariant, the “lengths” of the strings in Figure 2 are proportional to P^+ and therefore dependent on the Lorentz frame.) As we have already mentioned, GGRT constructed Lorentz generators for the free string and showed that they had the correct properties if $d = 26$. By applying these generators to the vertex (4.1), we have shown that the interacting-string theory is Lorentz invariant if $d = 26$ [30] [32].

Now let us see how we can obtain the result quoted in the introduction for the Born term of the string model. In that case the only horizontal lines in Figure 2 will be those from the external states, i. e., the string world sheet will have genus zero. Since the Neumann functions are conformally invariant, they can be evaluated by conformally transforming the string diagram onto the upper half plane, with the external particles transforming onto points on the real axis. The formula for doing so is a special case of the Schwarz-Christoffel transformation, namely

$$\rho = \sum_{r=1}^n \pi \alpha_r \ln(z - Z_r) \quad (4.7)$$

The variable $\rho (= \tau + i\sigma)$ is the co-ordinate on the string diagram, z the co-ordinate on the upper half-plane. The string lengths $\pi\alpha$ are considered positive for incoming strings, negative for outgoing strings. The r th string at $\tau = \pm\infty$ transforms to the point Z_r on the real axis. It is often convenient to take $Z_N = \infty$.

The Neumann function for the upper half-plane is simply

$$N(z, z') = \ln|z - z'| + \ln|z - z'^*| \quad (4.8)$$

(For closed strings, we transform the string diagram onto the whole plane. The strings at $\tau = \pm\infty$ transform to points anywhere in the plane, and the variables Z_r are complex. The second term on the right of (4.8) is absent.)

The momentum-independent factors of (4.6), i. e., the factors to the left of the exponential, may be treated in one of two ways. The simplest method is to make use of the proved Lorentz invariance of the amplitude in the critical dimension. We take the variables P^+ for all but two of the strings, one outgoing and one incoming, each at the bottom of Figure 2, to be equal to zero. Since the lengths of the strings are proportional to P^+ , this means that the string diagram consists of one long string with several infinitely short strings entering or exiting at the top. The vertex functions, and therefore the momentum-independent factor, are then easily shown to be unity. Thus, on transforming the integral (4.6) onto the upper half-plane, inserting the formula (4.8) for the Neumann functions, and replacing the $N - 3$ variables of integration τ_r by the corresponding real Z_r 's, we easily obtain the formula (1.1). (For closed strings the Z_r 's are integrated over the entire complex plane).

The foregoing analysis is very simple, but it cannot be extended to loops. An alternative method is to treat the string diagram in Figure 2 directly, without

using Lorentz invariance [33, p. 66]. We then have to examine the factor Δ in (4.6), the determinant of the Laplacian. For the upper half-plane, this factor is of course a constant; however, we require it for the string diagram and, since it requires regularization, it is *not* invariant under a conformal transformation. The change of Δ under such a transformation (known to physicists as the “conformal anomaly”) has been worked out by McKean and Singer [34] (see also Alvarez [35]). On applying their result to the conformal transformation from the string diagram to the upper half-plane, and calculating the Jacobian from the τ_r ’s to the Z_r ’s, which is not as simple as it was in the first method, we obtain equation(1.1), if and only if $d - 2 = 24$.

The calculation of the one-loop amplitude by functional integration involves new considerations, since the string-diagram world sheet is then conformally equivalent to an annulus instead of the upper half-plane. (The closed-string world sheet is conformally equivalent to a torus). For details of the calculation, we refer the reader to refs. [36], [37] [33, p.72].

Multiple-loop string diagrams for processes involving open and closed strings are conformally equivalent to Riemann surfaces of higher genus, possibly with holes if some of the strings are open. Such amplitudes are most easily treated by considering their analytic properties in the moduli space of Riemann surfaces of genus g . The analytic properties have been examined by Belavin and Knizhnik [38], who are mainly interested in the Polyakov formalism, to be outlined in the final section of this article, but their methods can also be used in the present context. More precisely, we take a conformal metric which is the product of an analytic function on the Riemann surface and its conjugate complex, as has been suggested by Sonoda [39]. Such a metric necessarily has $2(g - 1)$ zeros, which are of course singular points. It can then be shown that Δ , evaluated in this metric, is an analytic function in moduli space except for known singularities where the moduli space degenerates or where two zeroes of the metric coincide.

We should point out that the whole of this section is, strictly speaking, incorrect, since we have made an illegal Wick rotation in continuing to imaginary time. A Wick rotation cannot be made if there is an intermediate state with lower energy than the initial and final states, since the integral over the exponential $\exp\{-i(E_{init} - E_{int})t\}$ diverges. Normally such an intermediate state is subtracted out and treated explicitly, but such a procedure causes difficulties if applied to string models. In fact, all the amplitudes we have written here are real, and they diverge if the external energy in any channel is sufficiently large. We have already mentioned this fact in the introduction; we now see the reason for it from the point of view of string models. In the early work on dual models, the problem was treated by analytic continuation from sufficiently low energies. For closed-string amplitudes, it is not always possible to find a region where the energies in all channels are sufficiently low. Nevertheless, D’Hoker and Phong [40] have shown that the integral for the one-loop closed-string amplitude can be divided into several parts, each of which can be analytically continued. They thereby obtain a finite amplitude with the correct singularities. It would be preferable to perform the functional integration in such a way that the amplitude is finite and unitary as it stands, without the necessity of

analytic continuation. This can be done by undoing the illegal Wick rotation, as has been shown by Berera [41], again for the one-loop closed-string amplitude. It is not necessary to undo the entire Wick rotation; one need only consider the regions where the moduli space degenerates. One then obtains an amplitude which is finite and unitary.

5 Functional Integration for the RNS Model

In this section we shall summarize briefly how the functional integration of the previous section can be applied to the Ramond-Neveu-Schwarz model [42]. The advantage of the present treatment is that the Ramond (R) and Neveu-Schwarz (NS) models now appear much more directly as two sectors of a single model.

In addition to the commuting variables of the previous model, we have two anticommuting variables, S_1^i and S_2^i with (transverse) vector indices, corresponding to the anticommuting operators in the operator treatment of the RNS model. The Lagrangian will be the sum of two terms, one being the same as the Lagrangian (4.2) of the previous model, and the other being given by the formula

$$\mathcal{L}_2 = -\frac{1}{2\pi} \left\{ S_1 \left(\frac{\partial}{\partial\tau} + i \frac{\partial}{\partial\sigma} \right) S_1 + S_2 \left(\frac{\partial}{\partial\tau} - i \frac{\partial}{\partial\sigma} \right) S_2 \right\}. \quad (5.1)$$

The boundary conditions at the end of the string are either $S_1 = +S_2$ or $S_1 = -S_2$. For incoming fermions or outgoing anti-fermions (R) we have the plus sign at both ends; for incoming anti-fermions or outgoing fermions we have the minus sign at both ends, while for bosons (NS) we have different signs at the two ends. The boundary conditions are continuous along the top or bottom of the horizontal lines in Figure 2, but one sign changes into the other at each joining point. When expanding the S 's in normal modes b^i , we use the interval $-\pi\alpha \leq \sigma \leq \pi\alpha$, where $\pi\alpha$ is, as usual, the length of the string, and we take $S_2^i(-\sigma) = S_1^i(\sigma)$, $\sigma > 0$. Thus the bosons have half-integral mode numbers, the fermions integral mode numbers. With closed strings S_1 and S_2 change sign after one complete rotation for bosons and do not change sign for fermions. Thus the closed-string model has four sectors, NS-NS, NS-R, R-NS and R-R.

The vertex function is no longer given simply by the overlap integral, but there is an extra factor G given by the equation:

$$G = - \lim_{\sigma \rightarrow \sigma_1} i^{\pm 1/4} (\sigma - \sigma_1)^{3/4} S_1^i(\sigma) \left(\frac{\partial}{\partial\tau} + i \frac{\partial}{\partial\sigma} \right) X^i(\sigma) \quad (5.2)$$

The sign in the factor $i^{\pm 1/4}$ is plus or minus depending on whether the strings join or separate. The expression is evaluated at a point σ near the interaction point σ_1 . A factor G at each vertex will of course appear in the functional integrand. The factor G is necessary in order to prove Lorentz invariance, and in order that the functional integration give the same result as the operator formalism.

As in the Bose case, the functional integration can be performed in terms of Neumann functions. For tree-level amplitude, the result can be conformally

transformed onto the upper half-plane. Here we must bear in mind that the operators S^i , and therefore the Neumann functions involving them, have conformal weight $1/2$, i. e., on transforming to the z -plane we must include an extra factor $(\partial\rho/\partial z)^{-1/2}$ for S_1 and the complex conjugate of this factor for S_2 . We omit the details of the calculation. The result for scattering amplitudes of bosons is the same as that given by the operator calculation in the NS model. We can also calculate the scattering amplitude for fermion-antifermion scattering. This was the first such result as, in the operator formalism, amplitudes involving interacting Ramond fermions were much more difficult to treat than those involving bosons. The rather complicated calculations in the operator formalism were completed soon after by Schwarz and Wu [43], following work by Thorn, Corrigan and Olive; Olive and Scherk; and Brink, Olive, Rebbi and Scherk [44]. Their results agreed with those calculated by functional integration.

Berkovits [45] showed that the treatment of the RNS model outlined in this section was equivalent to a functional integration over a super-worldsheet. Superfields had previously been applied to the RNS model by Fairlie and Martin [46] and Brink and Winnberg [47], and also by Polyakov [48] in the formalism to be mentioned in the next section. The use of supersheets introduces a considerable simplification, since there is no longer an operator G at the interaction points.

6 Comparison with the Polyakov Formulation for Functional Integration

For completeness we shall now compare briefly the formulation of the last two sections with another formulation proposed later by Polyakov [48], even though it takes us beyond the time frame of this volume. (For a very much more detailed treatment of the Polyakov model, see ref. [31].) Instead of starting from the Nambu-Goto string, Polyakov starts from an action defined by Brink, DiVecchia and Howe [49] and by Deser and Zumino [50]. One introduces a general co-ordinate system on the world sheet, with co-ordinates (σ_a, σ_b) and metric g_{ab} ; the position of the point σ in d -dimensional space is denoted by $X^\mu(\sigma)$. The Lagrangian is then

$$\mathcal{L} = -\frac{1}{4\pi} g^{1/2} g_{ab} \left(\frac{\partial X^\mu}{\partial \sigma_a} \frac{\partial X_\mu}{\partial \sigma_b} \right). \quad (6.1)$$

To calculate the S -matrix with N external ground-state closed strings in g th order perturbation theory, Polyakov considers a Riemann surface of genus g with N punctures (which, as we have seen, is conformally equivalent to a surface with the strings going to $\pm\infty$). At each puncture there is a factor $\exp\{ip_r^\mu X_\mu(\sigma)\}$. The functional integrand is thus

$$\prod_{r=1}^N \int d^2\sigma_r g^{1/2}(\sigma_r) \exp \left\{ i \sum_r p_r^\mu X_\mu(\sigma_r) + \int d^2\sigma \mathcal{L}(\sigma) \right\}. \quad (6.2)$$

Polyakov now functionally integrates over the variables $X^\mu(\sigma)$ and $g_{ab}(\sigma)$. The integration over $X^\mu(\sigma)$ proceeds as before. To integrate over g_{ab} , we first notice that we can use an Einstein co-ordinate transformation to convert the metric to a covariantly constant metric $g\delta_{ab}$ (in Euclidean space). Due to invariance under such transformations, this integration can be reduced to a functional Faddeev-Popov determinant which will contain a conformal anomaly similar to the conformal anomaly in the determinant of the ordinary Laplacian mentioned in section 4. Unless $d = 26$, the conformal anomaly will appear as an extra field when integrating over the remaining g , and we obtain a string theory together with an extra Liouville field. This noncritical string theory has been studied to a certain extent, but not nearly as much as that with the critical dimension $d = 26$. For this dimension the conformal anomalies in the two functional determinants cancel, and the theory possesses a Weyl conformal invariance, i. e., an invariance under a conformal change of g without a compensating change in the co-ordinate system. “Most” of the integration over the remaining variable g will therefore simply be as integration over gauges and, in fact, no further Faddeev-Popov determinant is introduced. However, if the genus is not zero, not all Riemann surfaces of the same genus are conformally equivalent, and we are left with an integration over the variables characterizing the conformal class.

Not surprisingly, the Polyakov integral for genus zero reproduces the formula (1.1). Again, the amplitude is projectively invariant, so that the positions of three external particles must be fixed at arbitrary values.

The Polyakov formulation has the advantage over the light-cone formulation of section 4 of being relativistically covariant. Also, the integration variables characterizing the conformal classes of Riemann surfaces correspond to the parametrization of such classes by Bers [51]. Thus the Polyakov formulation can be related to the mathematical theory of Riemann surfaces. As the conformal anomaly cancels, one can avoid reference to the metric (though it may not always be advantageous to do so [52] [39]). On the other hand, the Polyakov formulation gives us only the S -matrix; one cannot go off shell, whereas the light-cone formulation gives us a complete quantum mechanical theory where we are not restricted to infinite times and can go off shell, albeit noncovariantly. Also, the Polyakov formulation treats different orders in perturbation theory separately and unitarity, even perturbative unitarity, is far from obvious. In fact two inequivalent theories with the same Born term cannot both be unitary. It is therefore important that D’Hoker and Giddings [53] have shown that the two formulations are in fact equivalent; they are different gauges of the same theory. D’Hoker and Giddings point out that it should be possible to prove unitarity directly in the Polyakov formulation (for a possible approach see [31, Chapter 9]), but such a proof appears to be considerably more complicated than in the light-cone formulation, where unitarity is manifest.

Thus, with the two gauges of the functional integration theory, we can more easily see different aspects of the system than we could with each gauge separately.

Polyakov also considered the RNS string, and most of the general remarks we have made apply to this case as well. Aoki, D’Hoker and Phong [54] have treated

the unitarity of the Polyakov RNS string by relating it to Berkovits' light-cone formulation with a super-worldsheet, mentioned in the previous section, which is manifestly unitary, but some problems remain with external fermions [55].

The RNS formalism has space-time supersymmetry after projecting out half of the states using the Gliozzi-Scherk-Olive (GSO) projection [56] (which we have not discussed in this article). However, it is not *manifestly* supersymmetric. Green and Schwarz [57], using the light-cone gauge, have given a formulation with manifest space-time supersymmetry. This makes it much easier to show that the principal divergence of the theory, namely the “dilaton” divergence, does not occur [58], as the absence of this divergence depends on space-time supersymmetry. The Green-Schwarz formalism requires operators at the interaction points of Figure 2, and calculations, especially of multi-loop amplitudes, are thereby made more difficult. At the cost of some complication, Berkovits [59] has given a manifestly Lorentz-covariant formulation of the Green-Schwarz theory, which includes the “ghosts” associated with the Faddeev-Popov determinant discussed above. His formulation thus possesses manifest Lorentz covariance and manifest space-time supersymmetry, and avoids the necessity of operators at the interacting points. As before, unitarity is established by comparing the theory with the light-cone formulation of the previous section.

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